Improper Integrals

Introduction # Riemann Stieltjes integral $\int_a^b f(x)\,d\alpha$ or Riemann integral $\int_a^b f\,dx$ is defined under the restriction that both f, α are defined and bounded on a finite interval $[a\ b]$. However the symbol $\int_a^b f\,d\alpha$ may sometimes have meaning (ie denote a number), even when f is not bounded or when either a or b or both are infinite. In such cases the symbol

$$\int_a^b f\,d\alpha$$

is called an improper or generalised or infinite integral. Thus the integrals with unbounded integrand or with unbounded interval of integration are improper integrals.

Note # For the sake of distinction an integral which is not improper will be called a proper integral:-

Improper Integral of First Kind

i) The integral $\int_a^b f\,d\alpha$ is called improper integral of 1st kind if the integrand remains bounded but integral Integration is unbounded.

Therefore $\int_1 \frac{1}{x^2}$ an

Def# Let f, α be defined on $[a, \infty)$. Suppose that $f \in R(\alpha; a, t) = R_\alpha[a\ t]$ for every $t \geq a$. Keeping a, f, α fixed define a function I on $[a\ \infty)$ as

$$I(t) = \int_a^t f(x)\, d\alpha(x) \qquad t \geq a.$$

The function $I(t)$ so defined is called an infinite integral (or an improper integral of 1st kind) and is denoted by $\int_a^\infty f\, d\alpha$.

The integral $\int_a^\infty f\, d\alpha$ is said to converge or said to exist if

$$\lim_{t \to \infty} I(t) = \lim_{t \to \infty} \int_a^t f\, d\alpha \text{ exists (finite)}$$

otherwise $\int_a^\infty f\, d\alpha$ is said to diverge or we say integral does not exist.

If $\lim_{t \to \infty} I(t)$ exists and equals A, then the number A is called value of the integral and we write

$$\int_a^\infty f(x)\, d\alpha(x) = A$$

Similarly we define the improper integral $\int_{-\infty}^b f\, d\alpha$ as

$$\int_{-\infty}^b f\, d\alpha = \lim_{t \to -\infty} \int_t^b f\, d\alpha$$

Q# Check the Convergence and Divergence of (1) $\int_1^\infty \frac{1}{x}\,dx$ (2) $\int_1^\infty \frac{1}{x^2}\,dx$.

Sol (1) $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} [\ln|x|]_1^t$$

$$= \lim_{t\to\infty} [\ln t - \ln 1]$$

$$= \lim_{t\to\infty} [\ln t] = \infty$$

$\Rightarrow \int_1^\infty \frac{1}{x}\,dx$ diverges.

(2) $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx$

$$= -\lim_{t\to\infty} \left[\frac{1}{x}\right]_1^t$$

$$= -\lim_{t\to\infty} \left[\frac{1}{t} - 1\right] = 1.$$

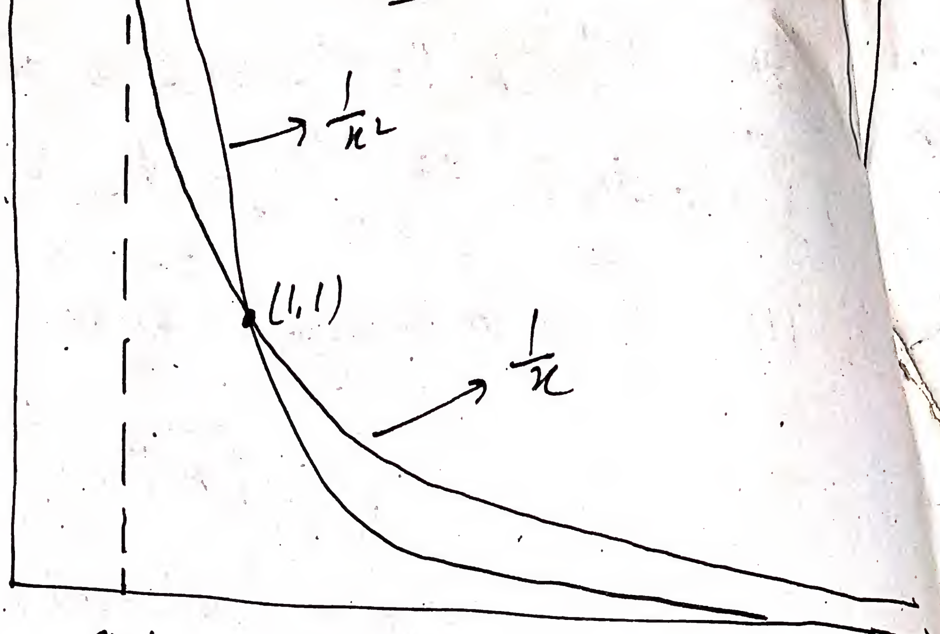
Discussion # We note that $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x^2}$

i.e both functions die as $x \to \infty$ but we also note that

$$\lim_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$\Rightarrow \frac{1}{x^2}$ dies faster than $\frac{1}{x}$ as $x \to \infty$

Therefore $\int_1^\infty \frac{1}{x^2}\,dx$ Converges.



$x=1$

We note that $\int_1^\infty x^2 dx$ diverges because $f(x) = x^2 \longrightarrow \infty$ as $x \to \infty$ i.e. does not die. Thus when a function does not die its integral does not converge but when a function dies as $x \to \infty$ we may expect its integral of type $\int_a^\infty f d\alpha$ to be convergent and we may equally expect it to be ~~converge~~ divergent as we have seen above in case of integrals $\int_1^\infty \frac{1}{x} dx$ & $\int_1^\infty \frac{1}{x^2} dx$

$$\boxed{\text{Integral } \int_{-\infty}^{+\infty} f(x)\, d\alpha(x)}$$

If for some $c \in (a\ \infty)$ $\int_{-\infty}^{c} f d\alpha$ and $\int_c^\infty f d\alpha$ are both convergent, then $\int_{-\infty}^{\infty} f d\alpha$ is cgt and its value is defined to be

$$\int_{-\infty}^{\infty} f d\alpha = \int_{-\infty}^{c} f d\alpha + \int_c^\infty f d\alpha$$

Cauchy Principal Value of $\int_{-\infty}^{\infty} f\,d\alpha$

Consider integral $\int_{-\infty}^{\infty} x\,dx$

The integral $\int_{1}^{\infty} x\,dx$ & $\int_{-\infty}^{1} x\,dx$ both diverge and hence the integral $\int_{-\infty}^{\infty} x\,dx$ diverges.

But

$$\lim_{c\to\infty} \int_{-c}^{c} x\,dx = 0.$$ It is called Cauchy principal value and it may exist even if the integral $\int_{-\infty}^{\infty} f\,d\alpha$ diverges as has been above.

Again $$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_{t}^{-1} \frac{1}{x^2} + \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx.$$

is divergent but

$$\lim_{c\to\infty} \int_{-c}^{c} \frac{1}{x^2}\,dx = -\lim_{c\to\infty} \left[\frac{1}{x}\right]_{-c}^{c}$$

$$= -\lim_{c\to\infty} \left[\frac{1}{c} + \frac{1}{c}\right] = 0$$

$$\Rightarrow \text{p.v} \int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = 0$$

But $\int_{-\infty}^{+\infty} f\,d\alpha$ Converges it Converges to its principal value i.e for a Convergent integral value of the integral is same as principal value.

Note # If we know the convergence of integral $\int_{-\infty}^{\infty} f\,dx$ in advance, we may find its value by finding the principal value.

Analogy Between Infinite Integral And Infinite Series

$$\int_a^{\infty} f\,dx \qquad\qquad \sum_{n=1}^{\infty} a_n$$

Here analogy is as

$$I(t) = \int_a^t f\,dx \quad \text{analogous to} \quad S_n = \sum_{k=1}^{n} a_k$$

partial integral — partial sum.

$\int$ is analogous to $\sum$

$f(x)$ is " " a_n

x corresponds to n

$x \downarrow$ varies continuously on $[a, \infty)$

$n \downarrow$ varies discretely on $\{1, 2, \ldots\}$

Improper Integral of 2nd Kind

If in the definite integral $\int_a^b f\,dx$, interval of integration is finite but f has one or more points of infinite discontinuity i.e f is not bounded on $[a, b]$, then $\int_a^b f\,dx$ is called an improper integral of 2nd kind.

e.g $\int_0^1 \frac{dx}{x}$, $\int_1^2 \frac{dx}{2-x}$

Improper Integral of 3rd Kind

If in definite integral $\int_a^b f d\alpha$, the interval is unbounded and f is also unbounded, then it is called improper integral of 3rd kind.

e.g $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}$

Convergence & Divergence of Improper Integral of 2nd Kind

(a) Convergence at Left End point

Let f be defined on $(a\ b]$ and integrable (RS or R) on $[t, b]$ $\forall t > a$ or on $[a+\epsilon, b]$, $\epsilon > 0$ or $\forall \epsilon$, $0 < \epsilon < b-a$, then $\int_a^b f d\alpha$ is defined by

$$\int_a^b f d\alpha = \lim_{\epsilon \to 0^+} I(\epsilon) = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f d\alpha$$

$$= \lim_{t \to a^+} \int_t^b f d\alpha.$$

If this limit exist and is equal to a real number A, then improper integral converges to A otherwise diverges.

Note If $\lim_{x \to a} f(x)$ exists but f is discontinuous at a, then $\int_a^b f d\alpha$ is considered as proper

and proper integral is always Convergent

→ If f is Continuous on $[a\ b]$ except that $f(c^+) \neq f(c^-)$, $a<c<b$ i.e f has a finite jump at c, then $\int_a^b f\,d\alpha$ is considered as proper.

(b) <u>Convergence at Right End Point</u> #

Let b be only point of infinite discontinuity and f is defined on (a,b), $f \in R(\alpha)$ on $[a,b)$ or on $[a,t]$ $\forall\, t<b$, $a \leq t < b$, then integral $\int_a^b f\,d\alpha$ defined as limit $\int_a^{b-\epsilon} f\,d\alpha$ as

$$\int_a^b f\,d\alpha = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f\,d\alpha \qquad 0<\epsilon<b-a$$

$$= \lim_{t \to b^-} \int_a^t f\,d\alpha \qquad 0<t<b-a, \quad t \in [a,b)$$

If this Limit exists, then the integral is cgt, otherwise dgt.

(c) <u>Convergence at Interior Point</u>

If an interior point c, $a<c<b$ is the only point of infinite discontinuty (i.e f is unbounded) at c, then

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Integral is cgt. if both integrals on R.H.S are Convergent otherwise is dgt.

(d) Convergence At Both End Points

If a & b are both points of infinite discontinuity, then for any c within the interval $[a, b]$

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

The integral exists if both integrals on R.H.S exist otherwise integral doesnot exist.

Example

Discuss the Convergence and divergence of integrals

(a) $\int_0^1 \frac{1}{x^p}\,dx$ (b) $\int_1^\infty \frac{1}{x^p}\,dx$ (c) $\int_0^\infty \frac{1}{x^p}$

Sol # (a) function $f(x) = \frac{1}{x^p}$ is continuous in $(0, 1]$ irrespective of the value of p but is undefined at $x=0$

Case 1 when $p<0$, f is bounded in $(0\ 1]$, so we can extend the definition to $x=0$ by setting the value of f to be 0 at $x=0$.

If $p=0$ f is identically 1 through out $[0\ 1]$.

Thus for $p \le 0$, f itself has a continuous extension to whole of $[0\ 1]$ and is Riemann integrable there

Case II # If $p>0$, $f(x) = \frac{1}{x^p}$ is unbounded at $x=0$ and integral is improper.

If $0<p<1$, then $1-p>0$ and

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x^p}\,dx = \lim_{t \to 0^+} \int_t^1 x^{-p}\,dx$$

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$$= \lim_{t \to 0^+} \left[\frac{x^{1-p}}{1-p} \right]_t^1$$

$$= \lim_{t \to 0^+} \left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p} \right)$$

$$= \frac{1}{1-p}$$

$\Rightarrow$ Integral Converges to $\frac{1}{1-p}$

<u>Case III</u> # If $p > 1$, $1-p < 0$ and

$$\int_0^1 \frac{1}{x^p} dx = \lim_{t \to 0^+} \int_t^1 x^{-p} dx$$

$$= \lim_{t \to 0^+} \left[\frac{1}{(1-p) x^{p-1}} \right]_t^1$$

$$= \lim_{t \to 0^+} \left(\frac{1}{1-p} - \frac{1}{(1-p) t^{p-1}} \right)$$

$$= \lim_{t \to 0^+} \left[\frac{1}{1-p} + \frac{1}{(p-1) t^{p-1}} \right]$$

$$= \infty$$

$\Rightarrow$ Improper integral diverges.

<u>Case IV</u> # For $p = 1$

$$\int_0^1 \frac{1}{x} dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x} dx$$

$$= \lim_{t \to 0^+} \left[\ln |x| \right]_t^1$$

$$= \lim_{t \to 0^+} \left[\ln 1 - \ln t \right]$$

$$= +\infty$$

$\Rightarrow$ Integral diverges for $p = 1$

Result $\Rightarrow \int_0^1 \frac{1}{x^p}\,dx$ is cgt if $p<1$
dgt if $p \geq 1$

(b) $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$

$= \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^t$

$= \lim_{t\to\infty} \left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right)$

$= \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ +\infty & \text{if } p<1 \end{cases}$

For $p=1$

$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty} [\ln|x|]_1^t = \lim_{t\to\infty} [\ln t - \ln 1]$

$= +\infty$

Result $\Rightarrow \int_1^\infty \frac{1}{x^p}\,dx$ is cgt if $p>1$
is dgt if $p \leq 1$.

(c) $\int_0^\infty \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{x^p}\,dx + \int_1^\infty \frac{1}{x^p}\,dx$

$I_1 + I_2$

For $p<0$ I_1 is cgt, I_2 is dgt to $+\infty$
For $p=0$ I_1 is cgt, I_2 is dgt to $+\infty$
For $0<p<1$ I_1 is cgt, I_2 is dgt to $+\infty$
For $p=1$ I_1 & I_2 both diverge to $+\infty$

For $p > 1$, I_1 is dgt. to $+\infty$ & I_2 is cgt.

$\Rightarrow$ For any arbitrary p one of the integral diverges to $+\infty$. Hence the integral diverges to $+\infty$ for all p.

Examples

Examine the Convergence and Divergence of

(1) $\int_0^\infty e^{-mx}\,dx \quad (m > 0)$ (2) $\int_a^\infty \frac{x}{1+x^2}\,dx$

(iii) $\int_0^\infty \sin x\,dx$ (4) $\int_0^\infty \frac{dx}{(1+x)^3}$

(5) $\int_0^\infty \frac{dx}{x^2+4a^2}$ (6) $\int_3^\infty \frac{dx}{(x-2)^2}$

(7) $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$ (8) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

(9) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$ (10) $\int_1^\infty \frac{x}{(1+2x)^3}\,dx$

Solutions

(1) $\int_0^\infty e^{-mx}\,dx = \lim_{t\to\infty} \int_0^t e^{-mx}\,dx$

$= \lim_{t\to\infty} \left[\frac{e^{-mx}}{-m}\right]_0^t$

$= -\frac{1}{m} \lim_{t\to\infty} \left[e^{-mt} - 1\right]$

$\therefore = -\frac{1}{m}[0-1] = \frac{1}{m}$, which is finite.

$\Rightarrow$ Integral converges

Result

$\int_0^\infty e^{-mx}\,dx$ Converges for every $m > 0$

(2) $\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \int_a^t \frac{x}{1+x^2}\,dx$

$= \lim_{t\to\infty} \left[\frac{1}{2}\log(1+x^2)\right]_a^t$

$= \lim_{t\to\infty} \frac{1}{2}\left[\log(1+t^2) - \log(1+a^2)\right]$

$= \infty$

$\Rightarrow$ Integral diverges.

(3) $\int_0^\infty \sin x\,dx = \lim_{t\to\infty} \int_0^t \sin x\,dx$

$= -\lim_{t\to\infty} \left[\cos x\right]_0^t$

$= -\lim_{t\to\infty} \left[\cos t - \cos 0\right]$

$= \lim_{t\to\infty} \left[\cos t - 1\right]$, which.

does not exist because $\cos t$ oscillates between -1 and 1

$\Rightarrow \int_0^\infty \sin x\,dx$ oscillates.

(4) # $\int_0^\infty \frac{dx}{(1+x)^3} = \lim_{t\to\infty} \int_0^t \frac{1}{(1+x)^3}\,dx$

$= \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(1+t)^2} - 1\right]$

$= -\frac{1}{2}(0-1) = \frac{1}{2}$

$\Rightarrow$ Integral Converges to $\frac{1}{2}$

(5) $\displaystyle\int_0^\infty \frac{dx}{x^2+4a^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+(2a)^2}$

$\displaystyle = \lim_{t\to\infty}\left[\frac{1}{2a}\tan^{-1}\frac{x}{2a}\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2a}\left[\tan^{-1}\frac{t}{2a} - \tan^{-1}0\right]$

$\displaystyle = \frac{1}{2a}\left[\frac{\pi}{2}\right] = \frac{\pi}{4a}$, which is finite

$\Rightarrow$ Integral Converges to $\frac{\pi}{4a}$

(6) $\displaystyle\int_0^\infty e^{2x}dx = \lim_{t\to\infty}\int_0^t e^{2x}dx.$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2}\left[e^{2x}\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\frac{1}{2}\left[e^{2t} - e^{0}\right]$

$= \infty$

$\Rightarrow$ Integral diverges to $+\infty$

Resut

Note that $\displaystyle\lim_{x\to\infty} e^{2x} = \infty$ i.e the integrand does not die as $x \longrightarrow \infty$ so integral diverges.

On the other hand $\displaystyle\int_0^\infty e^{-mx}$ Converges for all $m>0$. Here $\displaystyle\lim_{x\to\infty} e^{-mx} = 0 \quad \forall\, m>0$. So we may expect convergence which comes out.

$\star$ Knowledge non-negative and bounded is a great blessing of God.

(7) # $\int_3^\infty \frac{dx}{(x-2)^2} = \lim_{t\to\infty} \int_0^t (x-2)^{-2}\,dx$

$= \lim_{t\to\infty} \left[\frac{(x-2)^{-1}}{-1} \right]_0^t$

$= -\lim_{t\to\infty} \left[\frac{1}{t-2} - 1\right] = -(0-1) = 1$

$\Rightarrow$ Integral Converges.

(8) # $\int_0^\infty \frac{dx}{(1+x)^{2/3}} = \lim_{t\to\infty} \int_0^t (1+x)^{-2/3}\,dx$

$= \lim_{t\to\infty} 3 \left[(1+x)^{1/3}\right]_0^t$

$= 3 \lim_{t\to\infty} \left[(1+t)^{1/3} - 1\right]$

$= +\infty$, Divergent

(9) # $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \left[\sec^{-1} x\right]_{\sqrt{2}}^t$

$= \lim_{t\to\infty} \left[\sec^{-1} t - \sec^{-1}\sqrt{2}\right]$

$= \frac{\pi}{2} - \pi/4 = \frac{\pi}{4}$ which is finite.

$\Rightarrow$ Integral is cgt.

(10) # $\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \lim_{t\to\infty} \int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}\,dx$

$= \lim_{t\to\infty} \int_2^t \left[\frac{1}{x^2-1} + \frac{1}{x^2+1}\right] dx$

$= \lim_{t\to\infty} \left[\frac{1}{2} \ln\left(\frac{x-1}{x+1}\right) + \tan^{-1} x\right]_2^t$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\frac{t-1}{t+1} + \tan^{-1}t - \frac{1}{2}\log\frac{1}{3} - \tan^{-1}2\right]$$

$$= \frac{1}{2}\lim_{t\to\infty}\log\left(\frac{1-\frac{1}{t}}{1+1/t}\right) + \pi/2 + \frac{1}{2}\log 3 - \tan^{-1}2$$

$$= \frac{1}{2}\log 1 + \pi/2 + \frac{1}{2}\log 3 - \tan^{-1}2$$

$$= \frac{\pi}{2} + \frac{1}{2}\log 3 - \tan^{-1}2,$$ which is finite.

$\Rightarrow$ Integral Converges.

(11) # $\displaystyle\int_1^\infty \frac{x}{(1+2x)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}\,dx$

$$= \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}\,dx$$

$$= \lim_{t\to\infty}\int_1^t \left[\frac{1}{2}(1+2x)^{-2} - \frac{1}{2}(1+2x)^{-3}\right]dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\frac{(1+2x)^{-1}}{-1\times 2} - \frac{1}{2}\frac{(1+2x)^{-2}}{-2\times 2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2x)} + \frac{1}{8(1+2x)^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2t)} + \frac{1}{8(1+2t)^2} + \frac{1}{12} - \frac{1}{72}\right]$$

$$= 0 + 0 + \frac{1}{12} - \frac{1}{72} = \frac{5}{72}$$ which is finite.

$\Rightarrow$ Integral Converges.

Next Do yourself.

Examples#

Examine the Convergence or Divergence of Following integrals.

(1) $\int_1^\infty x e^{-x} dx$ (2) $\int_0^\infty x^2 e^{-x} dx$

(3) $\int_0^\infty x e^{-x^2} dx$ (4) $\int_0^\infty x^3 e^{-x^2} dx$

(5) $\int_0^\infty x \sin x \, dx$

Solution# (1) $\int_1^\infty x e^{-x} dx = \lim_{t\to\infty} \int_1^t x e^{-x} dx$

$= \lim_{t\to\infty} \left[-x e^{-x} - e^{-x}\right]_1^t = \lim_{t\to\infty} \left(-te^{-t} - e^{-t} + e^{-1} + e^{-1}\right)$

$= \lim_{t\to\infty} \left(-e^{-t} t\right) - \lim_{t\to\infty} e^{-t} + \frac{2}{e}$

$= 0 + 2/e = 2/e$ which is finite.

$\Rightarrow$ Integral Converges

(2) $\int_0^\infty x^2 e^{-x} dx = \lim_{t\to\infty} \int_0^t x^2 e^{-x} dx$

$= \lim_{t\to\infty} \left[-x^2 e^{-x} - 2x e^{-x} - 2 e^{-x}\right]_0^t$

$= 2$, which is finite

(3) $\int_0^\infty x e^{-x^2} dx = \lim_{t\to\infty} -\frac{1}{2} \int_0^t -2x e^{-x^2} dx$

$= -\frac{1}{2} \lim_{t\to\infty} \left[e^{-x^2}\right]_0^t$

$= -\frac{1}{2} \lim_{t\to\infty} \left(e^{-t^2} - e^0\right) = -\frac{1}{2}(0-1) = \frac{1}{2}$

(4)# $\int_0^\infty x^3 e^{-x^2} dx = \lim_{t\to\infty} \int_0^t x \cdot x^2 e^{-x^2} dx$

$= \lim_{t\to\infty} \left[-\frac{1}{2} x e^{-x^2} \Big|_0^t + \int_0^t \frac{1}{2} e^{-x^2} \cdot dx \right]$

Let $x^2 = z$ $\quad 2x\,dx = dz$

$x\,dx = \frac{1}{2} dz$

Limits When $x = 0 \quad z = 0$

$x = \infty \quad z = \infty$

$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2} \int_0^\infty z e^{-z} dz$

$= \frac{1}{2} \lim_{t\to\infty} \int_0^t z e^{-z} dz$

$= \frac{1}{2} \lim_{t\to\infty} \left[-z e^{-z} - e^{-z} \right]_0^t$

$= \frac{1}{2} \lim_{t\to\infty} \left[-t e^{-t} - e^{-t} - 0 + e^0 \right]$

$= \frac{1}{2} [0 + 0 + 1] = \frac{1}{2}$, which is Finite.

$\Rightarrow$ Integral Converges

(5) $\int_0^x x \sin x\, dx = \lim_{t\to\infty} \int_0^t x \sin x\, dx$

$= \lim_{t\to\infty} \left[-x \cos x + \sin x \right]_0^t$

$= \lim_{t\to\infty} (-t \cos t + \sin t)$

Which oscillates b/w $+\infty, -\infty$ because

$\cos t$ oscillates b/w -1 and $+1$ as $t\to\infty$

$\Rightarrow$ Integral oscillates infinitely.